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The renormalization-group approach to solving the equation of nonlinear transfer

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Abstract
With the help of an example of the equation of nonlinear transfer with power nonlinearity, it is shown how the requirement of functional self-similarity (renormalization invariance) enables one to construct a solution to the equation. Using this approach, the functional forms of the solution and additional conditions allowed are found by solving linear differential equations of the renormalization group, and the original nonlinear equation is only used for finding the numerical parameters of the solution (power exponents and coefficients). In addition, we present exact solutions to a transfer equation of a more general type that includes coordinates and space derivatives to arbitrary power.

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1. Introduction

The quasilinear parabolic equation of the second order is the basis of mathematical models for describing the nonlinear transfer phenomena. This equation is used in the study of processes such as nonlinear diffusion and heat conduction, transfer of turbulent energy, adiabatic filtration of fluids and gases in porous media, as well as a great variety of phenomena related to chemical kinetics, biochemistry, the problems of population and migration and so on. The universality of quasilinear parabolic equations is due to the fact that these equations describe the conservation laws of different physical quantities such as the number of particles, mass, energy, linear momentum and angular momentum, and others. The search for solutions is complicated due to the absence of the superposition principle widely used while solving linear problems.

The search for a solution can be made much easier by using the assumption of self-similarity. Commonly, in the search for self-similar solutions to nonlinear partial differential

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equations the symmetry properties of the equation under consideration, i.e. the properties of the invariance of the equation form under transformation of the function sought and its arguments, are studied. The search for these symmetries is carried out within the framework of the presently well-developed method of group analysis of differential equations (Ovsyannikov 1982, Olver 1986). The specification of initial or boundary conditions can break the symmetry, and hence for the existence of self-similar solutions it is required that additional conditions also possess the symmetry of the original equation. However, additional conditions are usually given at a fixed value of a coordinate (a boundary-value problem) or at a fixed value of time (an initial-value problem), and therefore it is not clear how one can verify the invariance of additional conditions under transformations of the symmetry group, which describe combined transformations of coordinate and time. Therefore, the question that should be posed is: what must be the functional form of additional conditions (initial or boundary) in order to leave the solution invariant under transformations of the symmetry group? Some methods were developed to tackle this problem (we may point out the most recent paper by Goard (2008)).

In the present paper, we propose an alternative method of solving this problem that is based on the renormalization-group arguments using the arbitrariness in setting additional conditions.

The family of solutions allowed by the symmetry group of the equation and additional conditions includes a set of numerical parameters of the symmetry transformations, and to choose a single-valued solution it is necessary to fix the values of these parameters by setting the values of the function and its derivatives at a certain point of the space of independent variables. However, such a point can be chosen arbitrarily and the result of this arbitrariness is a requirement of the invariance of the solution by varying the position of this point in combination with an appropriate change (renormalization) of numerical parameters of the problem. This property of the solution and additional conditions was named the renormalization invariance. Below, with the help of the example of the transfer equation with power nonlinearity of the flux, the way the property of renormalization invariance can be used when searching for self-similar solutions is demonstrated.

In section 2, we consider a one-dimensional transfer equation with power nonlinearity. To choose a single-valued solution, we set the values of the function and its time derivative at a certain spacetime point. Next, we exploit the dimensionality arguments to represent the solution in terms of the dimensionless function of the dimensionless variables. In section 3, the concept of renormalization-group invariance is explained and the functional and differential equations of the renormalization group are obtained. Section 4 is devoted to solving the renormalization-group equations. We find a functional form of the boundary condition as a solution to the linear differential equation and obtain a nonlinear ordinary differential equation for a function of a single variable. In a special limiting case, we obtain the solution with exponential time dependence. In the case of an initial-value problem, we find a functional form of the initial condition and obtain a nonlinear ordinary equation for another unknown function of a single variable. A solution to the equation of nonlinear transfer in a multi-dimensional radially symmetric case is presented in section 5. The solution has been obtained by solving the linear partial differential equations of the renormalization group. An equation of a more general form including spatial derivatives to arbitrary power is investigated in section 6 where the solution is obtained using the same technique.

2. The equation of nonlinear transfer and the dimensionality arguments when constructing the solution

The equation of transfer relates the time derivative of some scalar quantity and the divergence of its flux:

$$\frac{\partial u(\mathbf{r},t)}{\partial t} + \operatorname{div}\mathbf{Q}(u,\nabla u) = 0.$$

In a one-dimensional case and when the flux has the form

$$Q(x,t) = -\sigma u^m(x,t)\frac{\partial u(x,t)}{\partial x},$$

this equation may be written as

$$\frac{\partial u(x,t)}{\partial t} = \sigma\frac{\partial}{\partial x}u^m(x,t)\frac{\partial u(x,t)}{\partial x} \quad (m \text{ is arbitrary}). \tag{2.1}$$

This equation occurs in many problems of mathematical physics (Samarskii *et al* 1995); in particular, at $m > 1$ this equation is referred to as the equation of a porous medium and when $0 < m < 1$ it is called the equation of fast diffusion. The specific interest in these equations relates to the fact that under some special conditions they describe the so-called blow-up regimes, which were found in numerical calculations and were then supported experimentally (Samarskii *et al* 1995). One can find a detailed group analysis of equation (2.1) and a list of invariant (self-similar) solutions in the paper by Ovsyannikov (1959) (see also Pukhnachev 1995).

The simplicity of equation (2.1) enables one to readily recognize the symmetry properties of this equation with respect to the group of scale transformations $u \to \mu u, x \to \lambda x, t \to \lambda^2\mu^{-m}t$ and shifts of space and time variables $x \to x - x_1$ $t \to t - t_1$, where the parameters μ, λ, x_1 and t_1 are arbitrary. Careful group analysis (Ovsyannikov 1959) shows that there are no other symmetries except the case $m = -1/3$, which will be beyond the scope of our investigation. We note that the scale transformations specified by parameter μ relates to the ambiguity in a choice of units of measurement and while constructing the solution this corresponds to using the dimensionality arguments, whereas the ambiguity in a choice of parameter λ is a specific property of equation (2.1).

The existence of the symmetry pointed out means that if $u(x,t)$ is some solution to equation (2.1) obeying the additional conditions that preserve the symmetry properties of the original equation, the expression $\mu u(\lambda(x - x_1), \lambda^2\mu^{-m}(t - t_1))$ is also a solution to this equation if additional conditions are modified in a corresponding manner (the functional form of additional conditions is defined from the requirement that under symmetry transformations, this form remains unchanged). Thus, one has a family of solutions, which depend on the set of parameters λ, μ, x_1 and t_1. To choose a unique solution to the problem, it is necessary in addition to specify the set of parameters of symmetry group transformations. This can be realized by additionally specifying the values of the function and its derivative (spatial or temporal) at some spacetime point x_0, t_0 (hereinafter referred to as a 'normalization point'). In further presentation, we will use the following notation: values of the coordinate and time of the normalization point are denoted by $[x_0, t_0]$ and the value of the function at the normalization point is written as $u(x_0, t_0) = u(x,t)|_{[x_0,t_0]}$. According to this notation, the additional requirements for a choice of numerical parameters are written as

$$u(x,t)|_{[x_0,t_0]} = u_0, \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{[x_0,t_0]} = u_{0x}, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{[x_0,t_0]} = u_{0t}. \tag{2.2}$$

The second and third of relations (2.2) introduce some characteristic scales of length l_0 and time τ_0 according to formulae

$$l_0^{-1} = \left.\frac{\partial \ln u(x,t)}{\partial x}\right|_{[x_0,t_0]}, \qquad \tau_0^{-1} = \left.\frac{\partial \ln u(x,t)}{\partial t}\right|_{[x_0,t_0]}. \tag{2.3}$$

However, from the viewpoint of dimensionality arguments, the characteristic scales of length and time are not independent; they are connected by the relation $l_0 = \nu\sqrt{\sigma\tau_0 u_0^m}$, where ν is an arbitrary dimensionless parameter. The parameters u_0, τ_0 and l_0 introduced by setting additional conditions (2.2) have to be taken into account when carrying out the dimensionality analysis.

Below we will investigate a variant when in the normalization point the temporal derivative is given, and this corresponds to specifying the timescale τ_0 according to the second of relations (2.3). Choosing $x = 0, t = 0$ as the normalization point and using the dimensionality considerations, we find

$$u(x,t) = u_0 f\left(\frac{x}{\sqrt{\sigma\tau_0 u_0^m}}, \frac{t}{\tau_0}\right), \tag{2.4}$$

where $f(\xi,\eta)$ is a dimensionless function of dimensionless variables $\xi = x/\sqrt{\sigma\tau_0 u_0^m}$ and $\eta = t/\tau_0$ being subject to the normalization conditions

$$f(\xi,\eta)|_{[0,0]} = 1, \qquad \left.\frac{\partial f(\xi,\eta)}{\partial\eta}\right|_{[0,0]} = 1. \tag{2.5}$$

3. Renormalization invariance

Note that the choice of [0, 0] as the normalization point with relevant specification of the values of the function and one of its derivatives at this point using two parameters u_0 and $\tau_0 = u_0/u_{0t}$ is not unique, that is, one may take any other point $[x_1, t_1]$ as the normalization point and fix new (renormalized) values of the parameters u_1 and τ_1 by fitting those to the solution defined by specifying the parameters at the original normalization point [0, 0]. In doing so, the form of the solution remains invariant similar to the case when in order to choose a certain trajectory of a material point out of a family of trajectories it is required to give a value of point coordinate x_0 at time t_0, but instead of this it is possible to set the value of coordinate x_1 at another time t_1 as an initial condition, and in this case the form of trajectory remains invariant if the point x_1 lies in this trajectory (Teodorovich 2004), that is, we have the relation

$$x = X(t-t_0, x_0) = X(t-t_1, x_1), \qquad \text{where} \quad x_1 = X(t_1-t_0, x_0), \qquad X(0,x) = x.$$

A set of transformations corresponding to a shift of the normalization point in combination with a relevant change (renormalization) of numerical parameters set up at a new normalization point makes up a group named the renormalization group (RG), and the invariance of the solution form under RG transformations is referred to as functional self-similarity or renormgroup invariance (Shirkov 1982, Kovalev *et al* 1998).

The method of the renormalization group, which first originated in quantum-field theory, received wide application in various fields of mathematical physics (for example, see Kovalev *et al* 1998, Teodorovich 2004).

The property of functional self-similarity in application to the problem under consideration means that the following relationship should be fulfilled:

$$u(x,t) = u_0 f\left(\frac{x}{\sqrt{\sigma\tau_0 u_0^m}}, \frac{t}{\tau_0}\right) = u_1 f\left(\frac{x-x_1}{\sqrt{\sigma\tau_1 u_1^m}}, \frac{t-t_1}{\tau_1}\right), \tag{3.1}$$

where u_1 and τ_1 are defined in terms of values of the function $u(x,t)$ and its time derivative at a new normalization point.

To find renormalized parameters u_1 and τ_1, we put $x = x_1$ and $t = t_1$ in equation (2.4) and use normalization condition (2.5). As a result, we get the dependence of the renormalized parameters on the choice of the normalization point:

$$u_1 = u_0 f(\xi,\eta)_{[\xi_1,\eta_1]},$$
$$\tau_1^{-1} = \varphi(\xi,\eta)_{[\xi_1,\eta_1]}\tau_0^{-1}, \qquad \varphi(\xi,\eta) = \frac{\partial \ln f(\xi,\eta)}{\partial \eta}, \qquad \varphi(0,0) = 1. \tag{3.2}$$

Substitution of (3.2) into the condition of functional self-similarity (3.1) leads to the functional RG equation for the function $f(\xi,\eta)$:

$$f(\xi,\eta) = f(\xi_1,\eta_1) f\big((\xi-\xi_1)\sqrt{\varphi(\xi_1,\eta_1) f^{-m}(\xi_1,\eta_1)}, (\eta-\eta_1)\varphi(\xi_1,\eta_1)\big). \tag{3.3}$$

To obtain the RG differential equation, we differentiate equation (3.3) with respect to ξ_1 and next put $\xi_1 = \eta_1 = 0$. This leads to the equation

$$\alpha f(\xi,\eta) = \left[1 + \frac{1}{2}(\alpha m - \beta)\xi/2\right]\frac{\partial f(\xi,\eta)}{\partial \xi} - \beta\eta\frac{\partial f(\xi,\eta)}{\partial \eta}$$
$$\alpha = \left.\frac{\partial f(\xi,\eta)}{\partial \xi}\right|_{[0,0]}, \qquad \beta = \left.\frac{\partial \varphi(\xi,\eta)}{\partial \xi}\right|_{[0,0]}. \tag{3.4}$$

Similarly, differentiating (3.3) with respect to η_1 and putting $\xi_1 = \eta_1 = 0$, we find

$$f(\xi,\eta) = \frac{1}{2}(m-\gamma)\xi\frac{\partial f(\xi,\eta)}{\partial \xi} + (1-\gamma\eta)\frac{\partial f(\xi,\eta)}{\partial \eta}, \qquad \gamma = \left.\frac{\partial \varphi(\xi,\eta)}{\partial \eta}\right|_{[0,0]}. \tag{3.5}$$

The fact that when obtaining equations (3.3)–(3.5), the explicit form of the original equation (2.1) was not used should be noted; from this equation, only information on the dimensions of quantities in the equation and the invariance property under shifts of spacetime variables was used. This means that there are other equations possessing the same symmetry properties, for example $u_t = \sigma(u^{m+1})_{xx}$, $u_{tt} = \sigma(u^{2m+1})_{xxxx}$, etc. The requirement of the RG invariance of the solution expressed by equation (3.1) was added as some natural additional condition. Because of this, equations (3.3)–(3.5) cannot be treated as an equivalent of the original equation; these equations enable one to select a certain class of solutions and to reduce the number of independent variables by means of passing on to self-similar variables.

4. Solving RG equations

Now we pass on to the search for a solution to equation (3.5) which we will seek for in the form $f = \exp\Phi$; in this case $\Phi(0,0) = 0$ according to (2.5). Hence, the equation for Φ takes the form

$$1 = \frac{1}{2}(m-\gamma)\xi\frac{\partial \Phi(\xi,\eta)}{\partial \xi} + (1-\gamma\eta)\frac{\partial \Phi(\xi,\eta)}{\partial \eta}.$$

The solution to this equation can be represented as a sum of a particular solution to the inhomogeneous equation

$$\Phi_1(\eta) = -\frac{1}{\gamma}\ln(1-\gamma\eta)$$

and a general solution to the homogeneous equation that can be found by the method of characteristics:

$$\Phi_0(\xi,\eta) = \ln\psi(\zeta), \qquad \zeta = \xi(1-\gamma\eta)^{-(m-\gamma)/2\gamma}.$$

Thus, the solution to equation (3.5) turns out to be presented in the form

$$f(\xi,\eta) = (1-\gamma\eta)^{-1/\gamma}\psi(\zeta). \tag{4.1}$$

After substitution of (4.1) into the original equation (2.1), we get the equation for function $\psi(\zeta)$:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\psi^m\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} + \frac{m-\gamma}{2}\zeta\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} - \psi = 0 \qquad \psi(0) = 1, \qquad \psi'(0) = \alpha. \tag{4.2}$$

Solution (4.1) obtained is subject to boundary conditions of the form

$$u(0,t) = u_0(1+Bt)^b, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{[0,0]} = u_{0t} = bBu_0$$

and in terms of parameters B and b, which specify boundary conditions, this solution can be written as

$$u(x,t) = u_0(1+Bt)^b\psi\left(\sqrt{\frac{bB}{\sigma u_0^m}}x(1+Bt)^{-(mb+1)/2}\right). \tag{4.3}$$

The solution to form (4.1) has been obtained by Ovsyannikov (1959) and is presented in a monograph (Samarskii *et al* 1995) as an example of the solution, which describes the traveling wave for a 'power boundary regime'. Unlike Samarskii *et al* (1995), in the approach applied the boundary regime corresponding to (4.1) has been obtained from the requirement of renormalization invariance as a solution to the RG differential equation rather than as a lucky chosen (guessed) form of the boundary conditions allowed. If $B < 0$, solution (4.3) describes a 'blow-up regime for a 'power boundary condition' (Samarskii *et al* 1995).

Note that in the case $\gamma \to 0$, which in the language of the RG approach corresponds to the absence of renormalization of a characteristic timescale, by using the relation

$$(1-\gamma\eta)^{-1/\gamma}|_{\gamma\to 0} = \exp\{-\ln(1-\gamma\eta)/\gamma\}|_{\gamma\to 0} = \exp\{\eta\},$$

solution (4.1) takes the form

$$f(\xi,\eta) = e^{\eta}\psi(\xi\,\mathrm{e}^{-m\eta/2}). \tag{4.4}$$

Solution (4.4) is also contained in the paper by Ovsyannikov (1959) and is given in Samarskii *et al* (1995) as an example of a special choice of boundary behavior that allows a self-similar solution ('exponential boundary regime'). According to the results presented above, this regime proves to be a limiting case of the 'power boundary regime' and due to this fact the solutions of a similar form were named the limiting self-similar solutions (Barenblatt 1979). Such a solution is in the monograph (Barenblatt 1979), where while constructing a solution to equation (2.1) subject to the boundary condition $u(0,t) = u_0\mathrm{e}^{\nu t}$, the author paid attention to and used the fact that under a shift of the initial point of time by t_1 and carrying out the change $u_0 \to u_0\mathrm{e}^{\nu t_1}$ the solution has to hold its form, and this in fact reproduces the requirement of the RG invariance of the solution in a special case when there is no renormalization of the timescale under a shift of the normalization point along the time axis. The corresponding solution can be written in the form

$$u(x,t) = u_0\mathrm{e}^{\nu t}\psi\left(\sqrt{\frac{\nu}{\sigma u_0^m}}xt^{-m\nu t/2}\right), \qquad \nu = \frac{u_{0t}}{u_0}. \tag{4.5}$$

Self-similar solutions of another type can be obtained from equation (3.4) that corresponds to the requirement of invariance with respect to a shift of the normalization point along the coordinate axis. Repeating the procedure of constructing a solution to equation (3.5), we find

$$f(\xi,\eta) = (1+A\xi)^{1/A}\chi(\zeta), \qquad \zeta = \eta(1+A\xi)^{\beta/A}, \qquad A = \tfrac{1}{2}(m-\beta). \tag{4.6}$$

The function $\chi(\zeta)$ is a solution to the ordinary differential equation of the form

$$\frac{\mathrm{d}\chi}{\mathrm{d}\zeta} = \left[a\zeta^2\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + b\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} + c\right]\chi^{m+1}, \qquad \chi(0) = 1, \quad \chi'(0) = 1,$$

where the coefficients a, b and c are expressed in terms of parameters α, β and m.

Relationship (4.6) describes the solution to equation (2.1) that corresponds to initial conditions of the form $u(x, 0) = u_0(1 + Cx)^c$. In terms of the parameters C and c specifying the initial condition, this solution can be written as

$$u(x, t) = u_0(1 + Cx)^c\chi\left(\frac{u_{0t}t}{u_0}(1 + Cx)^{mc-2}\right). \tag{4.7}$$

In a special case $A = 0$ ($\beta = m$), which corresponds to the absence of timescale renormalization under a shift of the normalization point along the spatial axis, solution (4.6) takes the form

$$f(\xi, \eta) = \mathrm{e}^{\alpha\xi}\chi(\eta\,\mathrm{e}^{m\alpha\xi}). \tag{4.8}$$

To this case, the initial condition of the form $u(x, 0) = u_0\mathrm{e}^{x/l}$ (‘exponential boundary regime’) and the following form of the solution is assigned:

$$u(x, t) = u_0\mathrm{e}^{x/l}\chi\left(\frac{u_{0t}t}{u_0}\,\mathrm{e}^{mx/l}\right). \tag{4.9}$$

Solutions (4.7) and (4.9) were also obtained by Ovsyannikov (1959) as allowed self-similar solutions to equation (2.1) regardless of the choice of the initial condition form under which these solutions can be realized.

5. The multi-dimensional radially symmetric case

Consider the equation of the form

$$\frac{\partial u(r, t)}{\partial t} = \sigma\frac{1}{r^n}\frac{\partial}{\partial r}r^n u^m(r, t)\frac{\partial u(r, t)}{\partial r}, \tag{5.1}$$

which describes the nonlinear transfer of some scalar quantity in the space of dimension $(n + 1)$ when the distribution is a radially symmetric one. Within the framework of the commonly used approach, the solution to this equation is reproduced in a self-similar form (in dimensionless variables) $u(r, t) = t^\alpha\theta(r/t^\beta)$, and the power exponents α and β are defined from the requirement of existence of the conserved quantity (‘integral of motion’) $\int u(r, t)r^n\,\mathrm{d}r$ and consistency with the original equation. Consequently, one comes to a nonlinear ordinal differential equation for the function of a self-similar variable $\theta(\zeta)$, and solving this equation enables one to find a solution to equation (5.1). However, in many cases ‘the integral of motion’ pointed out diverges and the procedure described seems to be not sufficiently correct. For this reason, we present another way to construct the solution that is not based on the assumption of the existence of ‘integral of motion’.

First of all, we note that equation (5.1) does not possess the invariance with respect to a shift of the spatial argument r. The contraction of the symmetry group of equation (5.1) enables one to get more detailed information about the form of invariant solutions and initial and boundary conditions allowed; however, in this case, the above-outlined method has to be modified.

To choose a unique solution out of the family of solutions allowed by the symmetry group, assume that at the normalization point $[r_0, t_0]$ the values of the function and its spatial derivative are given by $u(r,t)|_{[r_0,t_0]} = u_0$, $\frac{\partial u(r,t)}{\partial r}|_{[r_0,t_0]} = u_{0r}$. As a result, in the theory, there arises a new parameter l_0 with the dimension of length defined by the relation

$$l_0^{-1} = \frac{u_{0r}}{u_0} = \frac{\partial \ln u(r,t)}{\partial r}\bigg|_{[r_0,t_0]}. \tag{5.2}$$

The dimensionality arguments lead to the following form of the solution:

$$u(r,t) = u_0 f\left(\frac{r}{r_0}, \frac{\sigma u_0^m (t-t_0)}{r_0^2}, \frac{l_0}{r_0}\right) \equiv u_0 f(\xi,\eta,g). \tag{5.3}$$

According to the normalization condition, the function $f(\xi,\eta,g)$ obeys the 'boundary conditions'

$$f(1,0,g) = 1, \qquad \frac{\partial \ln f(\xi,\eta,g)}{\partial \xi}\bigg|_{[1,0]} = \frac{1}{g}. \tag{5.4}$$

A requirement of renormalization invariance means that the solution form has to be unchanged under the variation of the normalization point $r_0 \to r_1, t_0 \to t_1$ in combination with the relevant renormalization of numerical parameters of the problem

$$u_0 \to u_1 = u_0 f(\lambda,\eta_1,g), \qquad u_{0r} \to u_{1r} = u_{0r}\varphi(\lambda,\eta_1,g),$$

where $\lambda = r_1/r_0$, $\eta_1 = \sigma u_0^m(t_1 - t_0)/r_0^2$, $\varphi(\xi,\eta,g) = \partial \ln f(\xi,\eta,g)/\partial\xi$.

From this, it follows that

$$u(r,t) = u_0 f\left(\frac{r}{r_0}, \frac{\sigma u_0^m (t-t_0)}{r_0^2}, \frac{l_0}{r_0}\right) = u_1 f\left(\frac{r}{r_1}, \frac{\sigma u_1^m (t-t_1)}{r_1^2}, \frac{l_1}{r_1}\right). \tag{5.5}$$

Relation (5.5) leads to the RG functional equation

$$f(\xi,\eta,g) = f(\lambda,\eta_1,g) f\left(\frac{\xi}{\lambda}, \frac{\eta-\eta_1}{\lambda^2} f^m(\lambda,\eta_1,g), \frac{1}{\lambda\varphi(\lambda,\eta_1,g)}\right). \tag{5.6}$$

Similar to (3.4)–(3.5) from (5.6), one can obtain two linear differential equations:

$$\left\{f_\xi - \xi\frac{\partial}{\partial\xi} - (2 - m f_\xi)\eta\frac{\partial}{\partial\eta} - (g + g^2\varphi_\xi)\frac{\partial}{\partial g}\right\} f(\xi,\eta,g) = 0 \tag{5.7a}$$

$$\left\{f_\eta - (1 - m f_\eta \eta)\frac{\partial}{\partial\eta} - (g^2\varphi_\eta)\frac{\partial}{\partial g}\right\} f(\xi,\eta,g) = 0$$

$$f_\xi = \frac{\partial f(\xi,0,g)}{\partial\xi}\bigg|_{\xi=1} = \frac{1}{g}, \qquad \varphi_\xi = \frac{\partial\varphi(\xi,0,g)}{\partial\xi}\bigg|_{\xi=1}, \tag{5.7b}$$

$$f_\eta = \frac{\partial f(1,\eta,g)}{\partial\eta}\bigg|_{\eta=0}, \qquad \varphi_\eta = \frac{\partial\varphi(1,\eta,g)}{\partial\eta}\bigg|_{\eta=0}.$$

Unfortunately, equations (5.7) include three g-dependent functions φ_ξ, f_η and φ_η, whose forms can be found from knowledge of function $f(\xi,\eta,g)$. Nevertheless, if one takes into account the fact that function $f(\xi,\eta,g)$ is a solution to the dimensionless form of equation (5.1)

$$\frac{\partial f}{\partial\eta} = \frac{1}{\xi^n}\frac{\partial}{\partial\xi}\xi^n f^m \frac{\partial f}{\partial\xi}, \tag{5.8}$$

one can find a relation between these functions, for example, by taking the function $\varphi_\xi(g)$ in the form of a polynomial of finite power and solving equation (5.7) at $\eta = 0$. (In the procedure of the RG method proposed by Bogolyubov and Shirkov (1955), these functions (RG functions) are defined using low-order perturbation theory with consequent substitution of the functions obtained in RG differential equations and solving these equations. Such a procedure corresponds to summing some infinite subsequence of total perturbation series (improved perturbation theory).) The solution to the equation for $f(\xi, 0, g)$ gives the form of initial conditions being in agreement with the request for renormalization invariance. Knowledge of the form of function $f(\xi, 0, g)$ enables one to find derivatives of this function with respect to ξ, and this gives a possibility of finding the required derivatives with respect to η using relations followed from (5.8) and then finding $f(\xi, \eta, g)$ by solving equations (5.7).

Following this program we assume $g + g^2\varphi_\xi = \alpha g + \beta$, where α and β are numerical parameters that have to be found later. Substitution of this expression into (5.7a) and a subsequent solution to the equation obtained enables one to find function $f(\xi, 0, g)$. According to the procedure given in the appendix, the solution has the form

$$f(\xi, 0, g) = \left\{1 + \frac{\beta}{\alpha g}(1 - \xi^\alpha)\right\}^{-1/\beta}. \tag{5.9}$$

We will seek for the solution $f(\xi, \eta, g)$ in the form

$$f(\xi, \eta, g) = \left\{\left(1 + \frac{\beta}{\alpha g}\right) A(\zeta) - \frac{\beta\xi^2}{\alpha g} B(\zeta)\right\}^{-1/\beta}, \tag{5.10}$$

where ζ is a solution to the equation of characteristics:

$$\frac{\mathrm{d}\eta}{2 - m/g} = \frac{\mathrm{d}g}{\alpha g + \beta}$$

and has the form

$$\zeta = \gamma\eta, \qquad \gamma = \frac{g^{m/\beta}}{(\alpha g + \beta)^{m/\beta + 2/\alpha}}. \tag{5.11}$$

The functions A and B obey initial conditions $A(0) = B(0) = 1$.

Substitution of (5.10) into a dimensionless version (5.8) of the original equation (5.1) gives that (5.10) will be a solution to this equation if

$$\alpha = 2, \qquad \beta = -m, \tag{5.12}$$

whereas the functions $A(\zeta)$ and $B(\zeta)$ are defined by relations

$$A(\zeta) = (1 - b\zeta)^{-a/b}, \quad B(\zeta) = (1 - b\zeta)^{-1}, \quad a = (n+1)m, \quad b = (n+1)m + 2. \tag{5.13}$$

Hence, the solution turns out to be represented in the form

$$f(\xi, \eta, g) = \left\{\left(1 - \frac{m}{2g}\right)\left(1 - \frac{b\eta}{g}\right)^{-a/b} + \frac{m\xi^2}{2g}\left(1 - \frac{b\eta}{g}\right)^{-1}\right\}^{1/m}. \tag{5.14}$$

It is easily verified that (5.14) is also a solution to the second RG differential equation (5.7b).

In terms of the original parameters of the problem u_0, u_{0r} and r_0 at $t_0 = 0$, the solution may be written as

$$\begin{aligned} u(r, t) &= u_0\left\{\left(1 - \frac{mu_{0r}r_0}{2u_0}\right)T(t)^{-a/b} + \frac{mu_{0r}}{2u_0 r_0}r^2 T(t)^{-1}\right\}^{1/m}, \\ u(r, 0) &= u_0\left\{1 + \frac{mu_{0r}}{2u_0 r_0}\left(r^2 - r_0^2\right)\right\}^{1/m} \\ T(t) &= 1 - \frac{b\sigma u_0^{m-1}u_{0r}}{2r_0}t. \end{aligned} \tag{5.15}$$

If $u_{0r} > 0$, solution (5.15) describes the blow-up regimes (Samarskii *et al* 1995) when the initially given distribution becomes singular at a certain point in a finite time interval. This solution is well known and can be found in Polyanin & Zaitsev (2002, item 1.1.10.7). Solution (5.15) allows the representation in a self-similar form:

$$u(r,t) = u_0\Gamma(t)^{-(n+1)}\left\{1 - \left(\frac{r_0}{R(0)}\right)^2 + \left(\frac{r}{R(0)\Gamma(t)}\right)^2\right\}^{1/m}, \tag{5.16}$$

where $\Gamma(t) = T(t)^{1/b}$, $R(0) = \sqrt{\frac{2u_0 r_0}{m|u_{0r}|}}$ and $T = (\bar{t} - t)/\bar{t}$ is the dimensionless time counted off the blow-up time $\bar{t} = 2r_0|u_{0r}|/b\sigma u_0^{m-1}$ in a reverse direction.

When $u_{0r} < 0$, the solution also turns out to be representable in a self-similar form:

$$u(r,t) = u_0\Gamma(t)^{-(n+1)}\left\{1 + \left(\frac{r_0}{R(0)}\right)^2 - \left(\frac{r}{R(0)\Gamma(t)}\right)^2\right\}^{1/m}$$
$$\frac{r}{R(0)\Gamma(t)} < \sqrt{1 + \left(\frac{r_0}{R(0)}\right)^2}, \tag{5.17}$$

where $T = (t - \bar{t})/\bar{t}$. This solution describes the propagation of a disturbance wave into an undisturbed medium at a finite speed of propagation; it was first obtained by Zel'dovich and Kompaneets (1950).

Thus, the realization of either the blow-up regime or the regime of propagation of the disturbance wave is defined by the sign of spatial derivative u_{0r} at the normalization point $[r_0, 0]$.

It should be noted the fact that while obtaining self-similar solutions (5.16) and (5.17) of the form

$$u(r,t) = C\Gamma(t)^{\beta}\Psi\left(\frac{r}{\Gamma(t)}\right) \tag{5.18}$$

the assumption of self-similarity was not used as a starting point and nor was one forced to solve the nonlinear differential equation for the function of a self-similar variable such as (4.3). The form of the solution was found by solving the first-order linear differential equations (RG differential equations).

6. An equation of a more general type

Without any modification, the method presented above appears to be applicable to the nonlinear equation of a more general type:

$$\frac{\partial u(x,t)}{\partial t} = \sigma\frac{1}{x^n}\frac{\partial}{\partial x}x^{n+p}u(x,t)^m\left|\frac{\partial u(x,t)}{\partial x}\right|^k \operatorname{sgn}\left(\frac{\partial u(x,t)}{\partial x}\right), \tag{6.1}$$

which is not considered in Polyanin & Zaitsev (2002).

The equation of such a form occurs in the problem on the propagation of seam hydrofracture in pumping a non-Newtonian fluid with a power rheological law.

Similar to the above-presented procedure, we will seek the solution in the form

$$u(x,t;x_0,t_0;u_0,u_{0x}) = u_0 f\left(\frac{x}{x_0}, \frac{\sigma u_0^{m+k-1}(t-t_0)}{x_0^{k+1-p}}, \frac{u_0}{u_{0x}x_0}\right), \tag{6.2}$$

where the dimensionless function of dimensionless variables $f(\xi, \eta, g)$ obeys the RG functional equation:

$$\begin{aligned} f(\xi, \eta, g) &= f(\lambda, \eta_1, g) f\left(\frac{\xi}{\lambda}, \frac{f^{m+k-1}(\lambda, \eta_1, g)}{\lambda^{k+1-p}}(\eta - \eta_1), g_1\right) \\ g_1^{-1} &= \lambda \frac{\partial}{\partial \xi} \ln f(\xi, \eta, g)|_{[\lambda, \eta_1]}. \end{aligned} \tag{6.3}$$

The solution to this equation is constructed following the above given scheme and in the self-similar form it can be written as

$$\begin{aligned} f(\xi, \eta, g) &= \frac{1}{\Gamma(\eta)^{n+1}} \left\{1 + \frac{\beta}{\alpha g} - \frac{\beta}{\alpha g}\left(\frac{\xi}{\Gamma(\eta)}\right)^{\alpha}\right\}^{-1/\beta} \\ \Gamma(\eta) &= \left(1 - \frac{c}{|g|^k} \operatorname{sgn} g \eta\right)^{1/c}; \end{aligned} \tag{6.4}$$

here $\alpha = (k + 1 - p)/k$, $\beta = -(m + k - 1)/k$ and $c = k[\alpha - (n + 1)\beta]$.

If $k = 1 - m$ (this case corresponds to $\beta = 0$), formula (6.4) leads to a limiting self-similar solution of an exponential form:

$$f(\xi, \eta, g) = \frac{1}{\Gamma(\eta)^{n+1}} \exp\left\{\frac{1}{\alpha g}\left[\left(\frac{\xi}{\Gamma(\eta)}\right)^{\alpha} - 1\right]\right\}. \tag{6.5}$$

In the above-obtained solution (6.4), the parameters u_0, u_{0x} and x_0 are not independent, and one of them can be excluded. When $g < 0$ the solution is positive definite in the finite domain $0 \leqslant x \leqslant L$ (such solutions are precisely that are of interest in the problem of transfer). As the numerical parameters of the problem we take $U = u(0, t_0)$ and L defined by the relation $u(L, t_0) = 0$. The corresponding mathematical treatment gives

$$\begin{aligned} u(x, t; U, L, t_0) &= \frac{U}{\Gamma(t)^{n+1}} \left\{1 - \left(\frac{x}{\Gamma(t) L}\right)^{\alpha}\right\}^{-1/\beta} \\ \Gamma(t) &= \left[1 + c\left|\frac{\alpha}{\beta}\right|^k \frac{U^{m+k-1}}{L^{k+1-p}}(t - t_0)\right]^{1/c}. \end{aligned} \tag{6.6}$$

By choosing various values of parameters n, p and k one can get solutions to some various equations, and the case $k = 1 - m$ when the solution is of an exponential form is an example. Similarly, if $p = (n+1)(m+k-1)+k+1$ (this case corresponds to $c = 0$) the time dependence turns out to be of an exponential form:

$$\Gamma(t) = \exp\{(t - t_0)/\tau\}, \qquad \tau^{-1} = (n + 1)^k (UL)^{m+k-1}.$$

Part of the solutions obtained is presented in Polyanin & Zaitsev (2002); however, even for these cases in the approach proposed it is uniquely defined to what initial (or boundary) conditions the corresponding solutions obey.

Note that at $n = p = 0$, the expression of the form

$$u(x, t) = \frac{U}{\Gamma(t)} \left\{1 - \left(\frac{|x| + a}{L\Gamma(t)}\right)^{\alpha}\right\}^{1/\beta}$$

is a solution to equation (6.1) by adding to the right-hand part the source term of the form $Q(t)\delta(x)$, where

$$Q(t) = -2a \frac{U^{m+k-1}}{L^{k+1}\Gamma(t)^{m+2k}} \left|\frac{m + k - 1}{k + 1}\right|^k u(0, t).$$

It should be pointed out that the above-described method of constructing the solution is applicable to some other problems, for example to the problem on transfer processes with the presence of chemical reactions (Teodorovich 1999) or to the case when the transfer coefficient is defined by the sign of the time derivative (Barenblatt's equation) (Barenblatt 1979). However, in all these cases one solves the problem of finding the power exponents of asymptotic behavior (anomalous dimensions or power exponents of incomplete self-similarity) (Goldenfeld *et al* 1990, Ginzburg *et al* 1991). In our analysis, we use the RG method for finding the exact solution to the nonlinear transfer equation.

7. Discussion

The part of solutions obtained within the framework of the RG approach are not new; they fall into the class of self-similar solutions found by Ovsyannikov (1959) and are presented in Polyanin & Zaitsev (2002). Nevertheless, the search for solutions within the framework of this approach radically differs from the commonly used approaches that are based on the Lie symmetry reduction of partial differential equations. Using the Lie symmetry method enables one to obtain ordinary differential equations such as equation (4.2) for function $\psi(\zeta)$ or for function $\chi(\zeta)$ in section 4. The problem of solving these equations and finding the functional forms of the initial or boundary conditions occurs to be beyond the scope of this method.

From the above-outlined presentation, one can see that principal distinctions between the Lie symmetry methods and renormalization-group approach, as well as a certain advantage of the RG approach, consist in the following.

(1) The assumption of the self-similar form of solution (5.18) is not taken as a starting point; this form is obtained as a result of solving the RG equations, which are linear differential equations of first order.
(2) The function of self-similar variables that defines the solution desired is a solution to the linear partial differential equation rather than a solution to the nonlinear ordinary differential equation as in the Lie symmetry reduction method.
(3) The functional forms of initial or boundary conditions allowed, which provide that the solution possesses the symmetry properties of the original equation, are found as solutions to the RG differential equations and not as suitably chosen forms. The numerical parameters of these forms are defined by setting the values of the function and its derivative at a chosen spacetime point (normalization point).
(4) Within the framework of the traditional approach briefly outlined in the beginning of section 5, the power exponents are found from the assumption of existence of the conserved quantity (integral of motion); however, for the solution obtained this integral diverges at a upper limit excluding the case of the solution by Zel'dovich and Kompaneets (1950) when the upper limit is finite. But at large negative values of parameter p in (6.1), this integral diverges at a lower limit and the relations for power exponents obtained appear to be ill justified. In the RG approach, the convergence of the integral of motion is not required.

The main purpose of this paper is to call attention to the renormalization-group method and to demonstrate its capability by a special example of the nonlinear transfer equation. As one can see, the RG method may be regarded as an addition to the Lie symmetry method. Namely, at the first stage one finds the symmetries of differential equation and invariants of the symmetry group to reduce the number of independent variables in the differential equation using the Lie symmetry method, and at the second stage the functional forms of the boundary conditions and the solution are found by solving the RG differential equations.

Appendix A.

The function $f(\xi, 0, g)$ is a solution to the equation

$$\left\{\frac{1}{g} - \xi\frac{\partial}{\partial\xi} - (2g-m)\frac{\partial}{\partial g}\right\} f(\xi, 0, g) = 0. \tag{A.1}$$

We will seek a solution in the form $f(\xi, 0, g) = \exp\{\Phi(\xi, g)\}$, where the function $\Phi(\xi, g)$ obeys the equation

$$\frac{1}{g} = \left\{\xi\frac{\partial}{\partial\xi} + (2g-m)\frac{\partial}{\partial g}\right\}\Phi(\xi, g). \tag{A.2}$$

We represent the solution as a sum of a particular solution to an inhomogeneous equation and a general solution to a homogeneous one $\Phi(\xi, g) = \Phi_1(g) + \Phi_0(\xi, g)$, where $\Phi_1(g)$ is a solution to the equation

$$\frac{1}{g} = (2g-m)\frac{\mathrm{d}\Phi_1(g)}{\mathrm{d}g}$$

and has the form

$$\Phi_1(g) = \frac{1}{m}\ln\left(1 - \frac{m}{2g}\right). \tag{A.3}$$

The general solution to the homogeneous equation is found using the method of characteristic and may be written as

$$\Phi_0(\xi, g) = \Psi(\zeta), \qquad \zeta = \ln\left(g - \frac{m}{2}\right) - \ln\xi^2 \equiv h(g) - \ln\xi^2. \tag{A.4}$$

Using the boundary condition $\Phi(1, g) = 0$, we obtain the form of function $\Phi_0(\xi, g)$ with the help of the relation $\Psi(h(g)) = -\Phi_1(g)$, from which it follows

$$\Psi(\zeta) = -\Phi_1(h^{-1}(\zeta)); \tag{A.5}$$

here, $h^{-1}(\zeta)$ is an inverse function with respect to function $h(g)$ defined by the relation $h^{-1}(h(g)) = g$. Using (A.3), we find that this function has the form $h^{-1}(\zeta) = \frac{m}{2} + \mathrm{e}^{\zeta}$.

Thus, we find

$$\Phi(\xi, g) = \frac{1}{m}\ln\left[1 + \frac{m}{2g}(\xi^2 - 1)\right]$$

and

$$f(\xi, 0, g) = \left\{1 + \frac{m}{2g}(\xi^2 - 1)\right\}^{1/m}. \tag{A.6}$$

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